

Complete Affine Kähler Manifolds

Fang Jia and An-Min Li ¹

Abstract. In this paper we prove that for a complete, connected and oriented Affine Kähler manifold (M, G) of dimension n , if it is affine Kähler Ricci flat or if the affine Kähler scalar curvature $S \equiv 0$, ($n \leq 5$), then the affine Kähler metric is flat.

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Introduction

An affine manifold is a manifold which can be covered by coordinate charts so that the coordinate transformations are given by invertible affine transformations. Let M be such affine manifold. We shall always assume that our coordinate systems are chosen as above and we call it affine coordinates. Let M be an affine manifold. An affine Kähler metric on M is a Riemannian metric on M such that locally, for affine coordinates (x_1, x_2, \dots, x_n) , there is a potential f such that

$$G_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

The pair (M, G) is called an affine Kähler manifold.

It is easy to see that the tangent bundle of an affine manifold is naturally a complex manifold. For each coordinate chart (x_1, x_2, \dots, x_n) , if we write a tangent vector of M as $\sum y_i \frac{\partial}{\partial x_i}$, then

$$z_i = x_i + \sqrt{-1}y_i, \quad i = 1, 2, \dots, n$$

are local holomorphic coordinates of TM . The affine Kähler metric naturally extends to be a Kähler metric of the complex manifold. The Ricci curvature and the scalar curvature of this Kähler metric are given respectively by

$$K_{ij} = - \sum \frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})),$$

$$S = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n f^{ij} \frac{\partial^2 \log \det (f_{kl})}{\partial x_i \partial x_j},$$

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where $f_{kl} = \frac{\partial^2 f}{\partial x_k \partial x_l}$. Following Cheng and Yau ([2]) we call K_{ij} and S the affine Kähler Ricci curvature and the affine Kähler scalar curvature of (M, G) respectively. We say that the affine Kähler metric G is Einstein if its Ricci tensor is a scalar multiple of the affine Kähler metric, that is

$$-\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = a f_{ij},$$

where a is a constant. In particular, if $a = 0$ then we call (M, G) affine Kähler Ricci flat.

Our main results can be stated as follows:

Theorem 1. *Let (M, G) be an n -dimensional complete, connected and oriented C^∞ affine Kähler Ricci flat manifold. Then the affine Kähler metric is flat.*

Theorem 2. *Let (M, G) be a complete, connected and oriented C^∞ affine Kähler manifold of dimension n . If the affine Kähler scalar curvature $S \equiv 0$, then, for $n \leq 5$, the affine Kähler metric is flat.*

As consequences we have

Theorem 3. *Let $f(x_1, \dots, x_n)$ be a smooth and strictly convex function defined in $\Omega \subset \mathbb{R}^n$. If the affine Kähler Ricci curvature is identically 0, and if the Calabi metric is complete, then f must be a quadratic polynomial.*

Theorem 4. *Let $f(x_1, \dots, x_n)$ be a smooth and strictly convex function defined in $\Omega \subset \mathbb{R}^n$. If the affine Kähler scalar curvature $S \equiv 0$, and if the Calabi metric is complete, then, for $n \leq 5$, f must be a quadratic polynomial.*

Remark. This is an unpublished paper that was finished in 2005. Since then, the formula of $\Delta\Phi$ (cf. Proposition 1) has been used frequently in various circumstances. Since it is used in our recent papers again (see [9, 10]), we decide to put this original version (with slight revision) on Arxiv.

1 Fundamental formulas

Let (M, G) be an affine Kähler manifold. Choose a local affine coordinate system (x_1, \dots, x_n) . Let $f(x)$ be a local potential function of G . Then f is locally strictly convex function and

$$G = \sum_{i,j} f_{ij} dx_i dx_j.$$

We recall some fundamental facts on the Riemannian manifold (M, G) (cf. [9]). The Levi-Civita connection is given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum f^{kl} f_{ijl}.$$

The Fubini-Pick tensor is

$$A_{ijk} = -\frac{1}{2} f_{ijk}.$$

Then the curvature tensor and the Ricci tensor are

$$\begin{aligned} R_{ijkl} &= \sum f^{mh} (A_{jkm} A_{hil} - A_{ikm} A_{hjl}) \\ R_{ik} &= \sum f^{mh} f^{jl} (A_{jkm} A_{hil} - A_{ikm} A_{hjl}). \end{aligned} \quad (1)$$

Let $\rho = [\det(f_{ij})]^{-\frac{1}{n+2}}$. Set

$$\Phi = \frac{\|\nabla \rho\|_G^2}{\rho^2} \quad (2)$$

$$4n(n-1)J = \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}. \quad (3)$$

It is easy to check that Φ and J are independent of the choice of the affine coordinate systems. Hence they are invariants globally defined on M . If $\Phi \equiv 0$ then $\rho = \text{constant}$. It is well known that (see [6])

$$\Delta J \geq 2(n+1)J^2. \quad (4)$$

Here and later the Laplacian and the covariant differentiation with respect to the metric G will be denoted by “ Δ ” and “ ∇ ” respectively.

2 Estimate for $\Delta \Phi$

In this section we calculate $\Delta \Phi$ for affine Kähler manifold with $S = 0$ and affine Kähler Ricci flat manifold.

Proposition 1. Let (M, G) be an n -dimensional, connected and oriented C^∞ affine Kähler Ricci flat manifold. Then the following estimate holds

$$\Delta \Phi \geq \frac{n}{n-1} \sum \frac{\|\nabla \Phi\|_G^2}{\Phi} + \frac{n^2 - 3n - 10}{2(n-1)} \langle \nabla \Phi, \nabla \log \rho \rangle_G + \frac{(n+2)^2}{n-1} \Phi^2.$$

Proof. Let $p \in M$ be any fixed point. Choose an affine coordinate neighborhood $\{U, \varphi\}$ with $p \in U$. We have:

$$-\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = (n+2) \left(\frac{\rho_{ij}}{\rho} - \frac{\rho_i}{\rho} \frac{\rho_j}{\rho} \right),$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\rho_{ij} = \frac{\partial^2 \rho}{\partial x_i \partial x_j}$. Noting that $-\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = 0$, we obtain

$$\frac{1}{\rho} \sum f^{ij} \rho_{ij} + \frac{n}{\rho^2} \sum f^{ij} \rho_i \rho_j - \frac{n+1}{\rho^2} \sum f^{ij} \rho_i \rho_j = 0,$$

where the matrix (f^{ij}) denotes the inverse matrix of the matrix (f_{ij}) . Then we have

$$\Delta \rho = \frac{n+4}{2} \frac{\|\nabla \rho\|_G^2}{\rho}. \quad (5)$$

We choose a local orthonormal frame field of the metric G on U . Then

$$\begin{aligned} \Phi &= \sum \frac{(\rho_{,j})^2}{\rho^2}, \quad \Phi_{,i} = 2 \sum \frac{\rho_{,j} \rho_{,ji}}{\rho^2} - 2 \rho_{,i} \sum \frac{(\rho_{,j})^2}{\rho^3}, \\ \Delta \Phi &= 2 \sum \frac{(\rho_{,ji})^2}{\rho^2} + 2 \sum \frac{\rho_{,j} \rho_{,jii}}{\rho^2} - 8 \sum \frac{\rho_{,j} \rho_{,i} \rho_{,ji}}{\rho^3} - (n-2) \frac{(\sum \rho_{,j}^2)^2}{\rho^4}, \end{aligned}$$

where we used (5). In the case $\Phi(p) = 0$, it is easy to get, at p ,

$$\Delta \Phi \geq 2 \sum \frac{(\rho_{,ij})^2}{\rho^2}. \quad (6)$$

Now we assume that $\Phi(p) \neq 0$. Choose a local orthonormal frame field of the metric g on U such that

$$\rho_{,1}(p) = \|\nabla \rho\|_G(p) > 0, \quad \rho_{,i}(p) = 0 \quad \text{for all } i > 1.$$

Then

$$\Delta \Phi = 2 \sum \frac{(\rho_{,ij})^2}{\rho^2} + 2 \sum \frac{\rho_{,j} \rho_{,jii}}{\rho^2} - 8 \frac{(\rho_{,1})^2 \rho_{,11}}{\rho^3} - (n-2) \frac{(\rho_{,1})^4}{\rho^4}. \quad (7)$$

Applying an elementary inequality

$$a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \geq \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{n-1}$$

and (5), we obtain

$$2 \frac{\sum (\rho_{,ij})^2}{\rho^2} \geq 2 \frac{(\rho_{,11})^2}{\rho^2} + 4 \frac{\sum_{i>1} (\rho_{,1i})^2}{\rho^2} + 2 \frac{\sum_{i>1} (\rho_{,ii})^2}{\rho^2}$$

$$\begin{aligned}
&\geq 2\frac{(\rho_{,11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} + \frac{2}{n-1}\frac{(\Delta\rho - \rho_{,11})^2}{\rho^2} \\
&\geq \frac{2n}{n-1}\frac{(\rho_{,11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{,1i})^2}{\rho^2} - 2\frac{n+4}{n-1}\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + \frac{(n+4)^2}{2(n-1)}\frac{(\rho_{,1})^4}{\rho^4}.
\end{aligned} \tag{8}$$

An application of the Ricci identity shows that

$$\frac{2}{\rho^2} \sum \rho_{,j} \rho_{,jii} = 2(n+4)\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} - (n+4)\frac{(\rho_{,1})^4}{\rho^4} + 2R_{11}\frac{(\rho_{,1})^2}{\rho^2}. \tag{9}$$

Substituting (8) and (9) into (7) we obtain

$$\begin{aligned}
\Delta\Phi &\geq \frac{2n}{n-1} \sum \frac{(\rho_{,11})^2}{\rho^2} + \left(2n - 2\frac{n+4}{n-1}\right) \frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} \\
&+ 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} + \left(\frac{(n+4)^2}{2(n-1)} - 2(n+1)\right) \frac{(\rho_{,1})^4}{\rho^4} + 4 \sum_{i>1} \frac{(\rho_{,1i})^2}{\rho^2}.
\end{aligned} \tag{10}$$

Note that

$$\sum \frac{(\Phi_{,i})^2}{\Phi} = 4 \sum \frac{(\rho_{,1i})^2}{\rho^2} - 8\frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} + 4\frac{(\rho_{,1})^4}{\rho^4}, \tag{11}$$

Then (10) and (11) together give us

$$\begin{aligned}
\Delta\Phi &\geq \frac{n}{2(n-1)} \sum \frac{(\Phi_{,i})^2}{\Phi} + \left(\frac{2n-8}{n-1} + 2n\right) \frac{(\rho_{,1})^2\rho_{,11}}{\rho^3} \\
&+ 2R_{11}\frac{(\rho_{,1})^2}{\rho^2} + \left(\frac{(n+4)^2}{2(n-1)} - 2(n+1) - \frac{2n}{n-1}\right) \frac{(\rho_{,1})^4}{\rho^4}.
\end{aligned} \tag{12}$$

From $\frac{\partial^2}{\partial x_i \partial x_j} \log \det(f_{kl}) = 0$ we easily obtain

$$\rho_{,ij} = \rho_{ij} + A_{ij1}\rho_{,1} = \frac{\rho_{,i}\rho_{,j}}{\rho} + A_{ij1}\rho_{,1}.$$

Thus we get

$$\Phi_{,i} = \frac{2\rho_{,1}\rho_{,1i}}{\rho^2} - 2\frac{\rho_{,i}(\rho_{,1})^2}{\rho^3} = 2A_{i11}\frac{(\rho_{,1})^2}{\rho^2}, \tag{13}$$

$$\frac{\sum (\Phi_{,i})^2}{\Phi} = 4 \sum (A_{i11})^2 \frac{(\rho_{,1})^2}{\rho^2}, \quad \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} = 2A_{111} \frac{(\rho_{,1})^3}{\rho^3}. \tag{14}$$

By the same argument of (8) we have

$$\sum (f_{ml1})^2 \geq (f_{111})^2 + 2 \sum_{i>1} (f_{i11})^2 + \sum_{i>1} (f_{ii1})^2$$

$$\begin{aligned}
&\geq (f_{111})^2 + 2 \sum_{i>1} (f_{i11})^2 + \frac{1}{n-1} \left(\sum f_{ii1} - f_{111} \right)^2 \\
&\geq \frac{n}{n-1} \sum (f_{i11})^2 - \frac{2}{n-1} f_{111} \sum f_{ii1} + \frac{1}{n-1} \left(\sum f_{ii1} \right)^2 \\
&= \frac{n}{n-1} \sum (f_{i11})^2 + \frac{2(n+2)}{n-1} f_{111} \frac{\rho_1}{\rho} + \frac{(n+2)^2 (\rho_1)^2}{n-1 \rho^2}.
\end{aligned} \tag{15}$$

Combing (1), (14) and (15) we have

$$2R_{11}(p) \frac{(\rho_1)^2}{\rho^2} \geq \frac{n}{2(n-1)} \sum \frac{(\Phi_{,i})^2}{\Phi} - \frac{(n+2)(n+1)}{2(n-1)} \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} + \frac{(n+2)^2 (\rho_1)^4}{2(n-1) \rho^4}. \tag{16}$$

Then

$$\Delta \Phi \geq \frac{n}{n-1} \sum \frac{(\Phi_{,i})^2}{\Phi} + \frac{n^2 - 3n - 10}{2(n-1)} \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} + \frac{(n+2)^2}{n-1} \Phi^2. \tag{17}$$

Proposition 2. Let (M, G) be an n -dimensional, connected and oriented C^∞ affine Kähler manifold with $S \equiv 0$. We have

$$\Delta \Phi \geq \frac{n}{2(n-1)} \frac{\|\nabla \Phi\|_G^2}{\Phi} + \frac{n^2 - 4}{n-1} \langle \nabla \Phi, \nabla \log \rho \rangle_G + \frac{(n+2)^2}{2} \left(\frac{1}{n-1} - \frac{n-1}{4n} \right) \Phi^2. \tag{18}$$

Proof. From the proof of Proposition 1 we see that the equality (12) remains hold. On the other hand

$$\begin{aligned}
2R_{11}(p) \frac{(\rho_1)^2}{\rho^2} &= \frac{1}{2} \sum (f_{kj1})^2 \frac{(\rho_1)^2}{\rho^2} + \frac{n+2}{2} f_{111} \frac{(\rho_1)^3}{\rho^3} \\
&\geq \frac{1}{2} \left[f_{111}^2 + \frac{1}{n-1} \left(f_{111} + (n+2) \frac{\rho_1}{\rho} \right)^2 \right] \frac{\rho_{,1}^2}{\rho^2} + \frac{n+2}{2} f_{111} \frac{\rho_{,1}^3}{\rho^3} \\
&\geq \frac{(n+2)^2}{2(n-1)} \Phi^2 - \frac{(n+2)^2 (n+1)^2}{8n(n-1)} \Phi^2 \geq -\frac{(n+2)^2 (n-1)}{8n} \frac{(\rho_1)^4}{\rho^4}.
\end{aligned}$$

This combined with (12) yields

$$\begin{aligned}
\Delta \Phi &\geq \frac{n}{2(n-1)} \sum \frac{(\Phi_{,i})^2}{\Phi} + \left(\frac{2n-8}{n-1} + 2n \right) \frac{(\rho_1)^2 \rho_{,11}}{\rho^3} \\
&\quad + 2R_{11} \frac{(\rho_1)^2}{\rho^2} + \left(\frac{(n+4)^2}{2(n-1)} - 2(n+1) - \frac{2n}{n-1} \right) \frac{(\rho_1)^4}{\rho^4} \\
&\geq \frac{n}{2(n-1)} \sum \frac{(\Phi_{,i})^2}{\Phi} + \frac{n^2 - 4}{n-1} \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} + \frac{(n+2)^2}{2} \left(\frac{1}{n-1} - \frac{n-1}{4n} \right) \frac{(\rho_1)^4}{\rho^4}.
\end{aligned}$$

3 Proof of Theorems

It is well known that an affine complete, parabolic affine hypersphere must be a quadratic polynomial. Using (4) and the same argument we can get

Lemma 1. *Let (M, G) be a complete, connected and oriented C^∞ affine Kähler manifold of dimension n . If $\Phi \equiv 0$, then any local potential function f of G must be a quadratic polynomial.*

Proof of Theorem 1. By Lemma 1 it suffices to prove that $\Phi \equiv 0$. Consider the function

$$F = (a^2 - r^2)^2 \Phi$$

defined on $B_a(p_0)$. Obviously, F attains its supremum at some interior point p^* of $B_a(p_0)$. Then, at p^* ,

$$\frac{\Phi_{,i}}{\Phi} - 2 \frac{(r^2)_{,i}}{a^2 - r^2} = 0. \quad (19)$$

$$\frac{\Delta \Phi}{\Phi} - \sum \frac{(\Phi_{,i})^2}{\Phi^2} - 2 \sum \frac{(r^2)_{,i}^2}{(a^2 - r^2)^2} - 2 \frac{\Delta(r^2)}{a^2 - r^2} \leq 0. \quad (20)$$

Inserting (19) into (20) we get

$$\frac{\Delta \Phi}{\Phi} \leq 24 \frac{r^2}{(a^2 - r^2)^2} + \frac{4}{a^2 - r^2} + 4 \frac{r \Delta r}{a^2 - r^2}. \quad (21)$$

Denote by $a^* = r(p_0, p^*)$. In the case $p^* \neq p_0$ we have $a^* > 0$. Let

$$B_{a^*}(p_0) = \{p \in M | r(p_0, p) \leq a^*\}.$$

By (17) we have

$$\max_{p \in B_{a^*}(p_0)} \Phi(p) = \max_{p \in \partial B_{a^*}(p_0)} \Phi(p).$$

On the other hand, we have $a^2 - r^2 = a^2 - a^{*2}$ on $\partial B_{a^*}(p_0)$, it follows that

$$\max_{p \in B_{a^*}(p_0)} \Phi(p) = \Phi(p^*).$$

Let $p \in B_{a^*}(p_0)$ be any point. Then from the definition of R_{ik} , we get

$$\begin{aligned} R_{ii}(p) &= \frac{1}{4} \sum f^{jl} f^{hm} (f_{hil} f_{mji} - f_{hii} f_{mjl}) \\ &\geq -\frac{(n+2)^2}{16} \Phi(p) \geq -\frac{(n+2)^2}{16} \Phi(p^*). \end{aligned}$$

Thus, by Laplacian comparison theorem (see [6] Appendix 2), we obtain

$$r\Delta r \leq (n-1) \left(1 + \frac{n+2}{4} \sqrt{\Phi(p^*)} \cdot r \right). \quad (22)$$

In the case $p^* = p_0$, we have $r(p_0, p^*) = 0$. Consequently, from (21) and (22), it follows that

$$\frac{\Delta \Phi}{\Phi} \leq \left(24 + \frac{(n-1)^2(n+2)^2}{4} \right) \frac{r^2}{(a^2 - r^2)^2} + \frac{4n}{a^2 - r^2} + \Phi. \quad (23)$$

On the other hand, by (17) we have

$$\frac{\Delta \Phi}{\Phi} \geq -\frac{(n^2 - 3n - 10)^2}{(n-1)^2} \frac{a^2}{(a^2 - r^2)^2} + \left(\frac{(n+2)^2}{n-1} - 1 \right) \Phi. \quad (24)$$

where we used (19). Inserting (24) into (23) we get

$$(a^2 - r^2)^2 \Phi \leq C_1(n) a^2,$$

where $C_1(n)$ is a constant depending only on n . Hence, at any interior point of $B_{\frac{a}{2}}(p_0)$, we have

$$\Phi \leq \frac{16C_1(n)}{9a^2}.$$

Let $a \rightarrow \infty$, then $\Phi \equiv 0$. We complete the proof of Theorem 1. ■

Applying a similar method and using the differential inequality (18) we can prove theorem 2.

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An-Min Li
 Department of Mathematics
 Sichuan University
 Chengdu, Sichuan
 P.R.CHina
 e-mail:math-li@yahoo.com.cn

Fang Jia
 Department of Mathematics
 Sichuan University
 Chengdu, Sichuan
 P.R.China
 e-mail:jiafangscu@yahoo.com.cn